International Journal of Engineering, Science and Mathematics
Vol. 10 Issue 10, October 2021,
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijmra.us, Email: editorijmie@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

A NEW CLASS OF OPEN SETS USING δ-PREOPEN SETS

Vidhyapriya P[1] and Sivakamasundari K[2]
[1] *Research Scholars,Department of Mathematics,*
[1]*E-mail: pvidhyapriya19@gmail.com*
[2]*Professor, Department of Mathematics,*
Avinashilingam Institute for Home Science and Higher Education for Women,
Coimbatore- 641043.

Abstract: The scope of this paper is to introduce a new class of sets called δP_S-open sets in topological spaces. This class of sets lies strictly between the classes of P_S-open and δ-preopen sets.

Keywords: δ-preopen, preopen, δ-open, Semi-open, Regular open, Regular semi-open

1. INTRODUCTION

In 1968 [26], the class of δ-open subsets of a topological space was first introduced by Velicko. This class of sets plays an important role in the study of various properties in topological spaces. Since then many authors used this class to define new classes of sets in topological spaces. In 1993, Raychaudhuri and Mukherjee [24] introduced and investigated a class of sets called δ-preopen.

Khalaf and Asaad [4] introduced a new concept called P_S-open sets in topological spaces. This class of sets lies strictly between the classes of δ-open and preopen sets. Combining the concepts of δ-preopen and P_S-open sets, a new class of sets called δP_S-open sets is introduced in this article. This class of sets lies between the classes of P_S-open and δ-preopen sets.The behaviour of δP_S-open sets in various spaces such as locally indiscrete, hyperconnected, extremally disconnected, semi-T_1, s-regular spaces are discussed and various interesting results are obtained.

2. PRELIMINARIES

Definition 2.1. A subset A of a space X is said to be

a) Preopen [19] if $A \subseteq Int(Cl(A))$
b) Semi-open [17] if $A \subseteq Cl(Int(A))$
c) Regular open [26] if $A = Int(Cl(A))$
d) Clopen if A is both open and closed
e) δ-open [27] if for each $x \in A$, there exists an open set G such that $x \in G \subseteq IntClG \subseteq A$
f) θ-open [28] if for each $x \in A$ there exists an open set G such that $x \in G \subseteq ClG \subseteq A$
g) δ-preopen [25] if $A \subseteq Int(\delta Cl(A))$
h) θ-semi-open [16] if for each $x \in A$, there exists an semi-open set G such that $x \in G \subseteq ClG \subseteq A$
i) semi-θ-open [7] if for each $x \in A$, there exists an semi-open set G such that $x \in G \subseteq sClG \subseteq A$

j) β-open [1] if A⊆ Cl (Int (Cl(A)))
k) η-open [10] if A is a union of δ-closed sets.
l) e*-open[12] if $A \subseteq Cl(Int(\delta - Cl(A))$
m) δ-semi-open[23] if $A \subseteq Cl(\delta - Int(A))$
n) α-open [20] if A ⊆ Int(Cl(Int(A)))
o) δ-semi-θ-open [29] if for each $x \in A$, there exists a δ-semiopen such that $x \in U \subseteq \delta$-$scl(U) \subseteq A$
p) δ-semiregular [29] if $A = \delta - sInt(\delta - sCl(A))$

- "The closure and interior of A with respect to X are denoted by Cl(A) and Int(A) respectively.
- The family of all preopen (resp. Semi-open, regular open, δ-open, θ-open, δ-preopen, θ-semiopen, β-open, η-open, e*-open, δ-semiopen, α-open, δ-semi-θ-open, δ-semiregular open) subsets of X is denoted by PO(X) (resp. SO(X), RO(X), δO(X), θO(X), δPO(X), θSO(X), SθO(X),βO(X), ηO(X),e*O(X), δSO(X), αO(X), δSθO(X), δSRO(X)
- The complement of a preopen (resp. Semi-open, regular open, δ-open, θ-open, δ-preopen, θ-semiopen,semi-θ-open, β-open, η-open, e*-open, δ-semiopen, α-open, δ-semi-θ-open, δ-semiregular open) is said to be preclosed [13] (resp. Semi-closed, regular closed, δ-closed, θ-closed, δ-preclosed, θ-semiclosed, semi-θ-closed,β-closed, η-closed, regular semiclosed, e*-closed, δ-semiclosed, α-closed, δ-semi-θ-closed, δ-semiregular closed).
- The family of all preclosed (resp. resp. Semi-closed, regular closed, δ-closed, θ-closed, δ-preclosed, θ-semiclosed, β-closed, η-closed, regular semiclosed, e*-closed, δ-semiclosed, α-closed, δ-semi-θ-closed, δ-semiregular closed) subsets of X is denoted by PC(X) (resp. SC(X), RC(X), δC(X), θC(X), δPC(X), θSC(X), SθC(X),βC(X), ηC(X), e*C(X), δSC(X), αC(X), δSθC(X), δSRC(X)).
- The family of all α-open sets in a topological space (X, τ) is a topology on X finer than τ denoted by τ_α
- The intersection of particular class of closed sets ofXcontaining A is called the corresponding closure of A.
- The union of particular class of open sets of X contained in A is called the corresponding interior of A.

Definition 2.2 [21].A subset A of a space X is said to be preregular if A is both preopen and preclosed.

Definition 2.3 [3]. A space X is s-regular if for each x∈X and each open set G containing x, there exists a semi-open set H such that x∈ H ⊆sClH ⊆G.

Definition 2.4 [18]. A space X is called semi-T_1 if for each pair of distinct points x, y in X, there exists a pair of semi-open sets, one containing x but not y and the other containing y but not x.

Theorem 2.5 [18]. A space X is semi-T_1 if for any point x ∈X, the singleton set {x} is semi-closed.

Definition 2.6 [7]. A space (X,τ) is said to be extremally disconnected if ClU ∈τ for every U ∈τ.

Theorem 2.7 [14]. A space X is extremally disconnected if and only if RO(X) = RC(X).

Theorem 2.8 [28]. A space X is extremally disconnected if and only if δO(X) = θSO(X).

Definition 2.9 [9]. A space X is said to be hyperconnected if every non-empty open subset of X is dense.

Lemma 2.10 [8]. A space (X,τ) is hyperconnected if and only if $RO(X) = \{X, \phi\}$.

Definition 2.11 [9]. A space X is called locally indiscrete if every open subset of X is closed.

Lemma 2.12. If X is locally indiscrete space, then

a) Each semi-open subset of X is closed and
b) Each semi-closed subset of X is open.

Theorem 2.13 [3]. Let (Y,τ_Y) be a subspace of a space (X,τ). Then, the following statements are true:

a) If A∈ PO(X,τ) and A⊆ Y, then A∈ PO(Y, τ_Y).
b) If F ∈ SC(X,τ) and F ⊆ Y, then F ∈ SC(Y, τ_Y).
c) If F ∈ SC(Y, τ_Y) and Y ∈ SC(X,τ), then F ∈ SC(X,τ).

Theorem 2.14 [25]. In a topological space (X,τ), if A ϵ δPO(X), B ϵ δO(X) then $A \cap B \in \delta PO(X)$.

Lemma 2.15. If Y is an open subspace of a space X and F ∈ SC(X), then F ∩ Y ∈ SC(Y).

Lemma 2.16 [15]. Let A be a subset of a space (X,τ). Then A∈ PO(X,τ) if and only if sClA = IntClA.

Theorem 2.17. Let A be a subset of a topological space (X,τ). Then, we have:

a) If A∈ SO(X), then pClA = ClA [9].
b) If A∈ βO(X), then $Cl_\delta A$ = ClA [24].
c) If A∈ βO(X), then αClX = ClX [2].

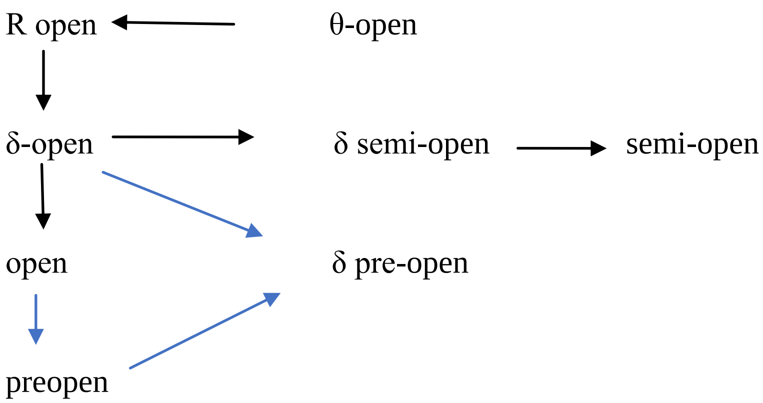
Theorem 2.18[29]. Let A, Y be subsets of a topological space (X,τ) and let A⊆ Y ϵ δPO(X). Then A ϵ δPO(X) if and only if A ϵ δPO(Y).

Definition 2.19[4]. A subset A of a space X is called P_S-open if for each x ϵA ϵ PO(X), there exists a semi-closed set F such that x ϵ F ⊆A. The family of all P_S-open sets of a topological space(X,τ) is denoted by $P_SO(X,\tau)$ or $P_SO(X)$.

Lemma 2.20[6]. Let A be a subset of a topological space (X, τ). Then if A is preopen in (X,τ), then it is δ-preopen in (X, τ).

International Journal of Engineering, Science and Mathematics
Vol. 10 Issue 10, October 2021,
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijmra.us, Email: editorijmie@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

Remark 2.21. From the following figure we have:



Lemma 2.22. Each clopen is regular open.
Lemma 2.23[25]. Union of δ-preopen sets is δ-preopen.
Lemma 2.24[25]. In a topological space (X,τ), if $A \in \delta PO(X), B \in aO(X)$, then $A \cap B \in \delta PO(X)$
Lemma 2.25[4]. Every δ-open set is P_S-open set.
Lemma 2.26[6]. If A is preopen then A is δ-preopen in (X,τ).
Lemma 2.27[29]. Let A, Y be subsets of a topological space (X,τ) and let $A \subseteq Y \in \delta PO$. Then $A \in \delta PO(X)$ if and only if $A \in \delta PO(Y)$.
Lemma 2.28[29]. Let A, B be subsets of topological space (X,τ). If $A \in \delta SO(X)$ and $B \in \delta PO(X)$, then $A \cap B \in \delta PO(A)$.

3. δP_S-Open Sets

Definition 3.1. A δ- preopen subset A of a space X is called a δP_S-open set if for each x ϵ A, there exists a semi-closed set F such that x ϵ F⊆A.

The family of all δP_S-open subsets of a topological space (X, τ) is denoted by $\delta P_S O(X, \tau)$ or $\delta P_S O(X)$.

Proposition 3.2. A subset A of a space X is δP_S-open if and only if A is a δ-preopen set and A is a union of semi-closed sets.

Proof: From the definition 3.1, a δP_S-open subset A of X is a δ preopen subset.

For every x ϵ A there exists a semi-closed set F_x such that $x \in F_x \subseteq A$

Hence$A = \cup_{x \in A}\{x\} \subseteq \cup F_x \subseteq A$, which will imply$A = \cup_{x \in A} F_x$,a union of semi-closed sets.

Remark 3.3. A δ-preopen set need not be a δP_S-open set. This can be seen from the following example.

Example 3.4. Let X={a,b,c} withthe topology τ = {X,φ,{a}}. Then δPO(X) = P(X) and SC(X) ={X,φ,{b},{c},{b,c}] and $\delta P_S O(X)$ ={X,φ,{b},{c},{b,c}}. Then {a} ϵ δPO(X), but {a}∉ $\delta P_S O(X)$.

International Journal of Engineering, Science and Mathematics
Vol. 10 Issue 10, October 2021,
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijmra.us, Email: editorijmie@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

Remark 3.5. Union of semi-closed sets need not be a δP_S open set.

Example 3.6. Let X = {a,b,c,d} withthe topology τ = {X,φ,{a},{c},{a,b},{a,c},{a,b,c}, {a,c,d}}. Then SC(X) ={X,φ,{b},{c},{d},{a,b},{b,c},{b,d},{c,d},{a,b,d},{b,c,d}} and $\delta P_S O(X)$ = {X,φ,{c},{b,c},{a,b,c},{b,c,d}}. Then {b},{d} ϵ SC(X) , but their union{b,d} ∉ $\delta P_S O(X)$.

Proposition 3.7. Any union of δP_S-open sets is a δP_S-open set.

Proof: Let$\{A_\alpha\}$be a collection of δP_S-open sets. Consider A=∪ A_α.

A is a δ-preopen set from the Lemma 2.23.

For every$x \in A, x \in A_\alpha$,for some α and sinceA_α is a δP_S-open set, there exists a semi-closed set F_x such that $x \in A_\alpha \subseteq F_x \subseteq \cup A_\alpha = A \therefore x \in F_x \subseteq A$

Thus,A is a δP_S-open set.

The following example shows that the intersection of two δP_S–open sets need not be δP_S-open set in general.

Example 3.8. Let X = (0,1). If A is the set of rational numbers in X and B is the set of irrational numbers in X together with the singleton set {1/2}. Then A ϵ PO(X) δPO(X). Since X is a T_1-space, every singleton set is closed and hence is semi-closed, then A ϵ $\delta P_S O(X)$ and B ϵ PO(X) SC(X) δPO(X) SC(X), then B ϵ $\delta P_S O(X)$. But A∩ B = {1/2} ∉ $\delta P_S O(X)$.

From the above example we notice that the family of all δP_S-open sets need not be a topology on X.

Proposition 3.9. If A and B are δP_S-open subsets of a topological space (X, τ) and if the family of all δ-preopen sets in X forms a topology on X, then $A \cap B$ is a δP_S-open set and hence the family of δP_S-open setforms a topology on X.

Definition 3.10[24]. A space (X, τ) is said to have the property P if the closure is preserved under finite intersection or equivalently, if the closure of intersection of any two subsets equals the intersection of their closures.

From the above definition Paul and Bhattacharyya [24] pointed out the following remark:

Remark 3.11. If a space X has the property P, then the intersection of any two preopen sets is preopen, as a consequence of this, PO (X, τ) is a topology for X and it is finer than τ.

Definition 3.12. If (X, τ) is said to have property P' if the δ-closure is preserved under finite intersection or equivalently, if the δ-closure of intersection of any two subsets equals the intersection of their δ-closures.

Lemma 3.13. If a space X has the property P', then the intersection of any two δ-preopen sets is δ-preopen, as a consequence of this, δPO(X, τ) is a topology for X and it is finer than τ.

Proof: Let A and B be δ-preopen subsets

$\therefore A \subseteq Int(\delta Cl\, A)$ and $B \subseteq Int\,(\delta Cl\, B)$

$$\Rightarrow A \cap B \subseteq Int\,(\delta\, Cl\, A) \cap Int\,(\delta\, Cl\, B)$$

$$\subseteq Int\,(\delta\, Cl\, A \cap \delta Cl\, B)$$

$\subseteq Int(\delta Cl(A \cap B))$ [∵ X has property P']

$\Rightarrow A \cap B$ is δ-preopen.

International Journal of Engineering, Science and Mathematics
Vol. 10 Issue 10, October 2021,
ISSN: 2320-0294 Impact Factor: 6.765
Journal Homepage: http://www.ijmra.us, Email: editorijmie@gmail.com
Double-Blind Peer Reviewed Refereed Open Access International Journal - Included in the International Serial Directories Indexed & Listed at: Ulrich's Periodicals Directory ©, U.S.A., Open J-Gage as well as in Cabell's Directories of Publishing Opportunities, U.S.A

Proposition 3.14. If (X,τ) possesses property P' mentioned in definition 3.12, then $\delta P_S O(X,\tau)$ forms a topology.

Proof: By Proposition 3.7, arbitrary union of δP_S-open sets is δP_S-open.

Let A and B are δP_S-open sets, Then A and B are δ-preopen and $A \cap B$ is δ-preopen from Lemma 3.13.

For each $x \in A \cap B$ there exist semi closed sets F_A & F_B such that

$$x \in F_A \subseteq A \text{ and } x \in F_B \subseteq B$$

$\Longrightarrow x \in F_A \cap F_B \subseteq A \cap B \Rightarrow x \in F \subseteq A \cap B$, where F is semi-closed. [put $F = F_A \cap F_B$]

Hence $A \cap B \in \delta P_S O(X)$. ∴ $A \cap B$ is $\delta P_S O$ (X,τ) forms a topology.

□ $\delta P_S O$ (X, τ_S) forms a topology.

Proposition 3.15. A subset A of a space (X, τ) is δP_S-open if and only if for each x ∈ A, there exists an δP_S-open set B such that x ∈ B ⊆ A.

Proof: If A is an δP_S-open subset in the space (X, τ), then for each x ∈ A, putting A = B, which is δP_S-open containing x such that x ∈ B ⊆ A. Conversely. Suppose that for each x ∈ A, there exists a δP_S-open set B such that x ∈ B ⊆ A. So, A = ∪Bγ where Bγ ∈ $\delta P_S O(X)$ for each γ. Therefore, by Proposition 3.7, A is δP_S-open.

Proposition 3.16. Let X be a topological space, and A, B ⊆ X. If A ∈ $\delta P_S O(X)$ and B is both α-open and semi-closed, then A ∩ B ∈ $\delta P_S O(X)$.

Proof: Let $A \in \delta P_S O(X)$ and B be α-open, then A is δP_S-open by definition 3.1. Then by Lemma2.24[29] $A \cap B \in \delta PO(X)$. Now let $x \in A$ and there exists a s-closed set F such that $x \in F \subseteq A$. Since B is s-closed, $F \cap B$ is sclosed and hence $x \in F \cap B \subseteq A \cap B$.

Thus $A \cap B$ is δP_S-open in X.

Proposition 3.17. If a space X is semi-T_1, then $\delta P_S O(X) = \delta PO(X)$.

Proof: Let $A \subseteq X$ and $A \in \delta PO(X)$. If $A = \varphi$, then $A \in \delta P_S O(X)$. If $A \neq \varphi$, then for each x ϵ A, by Theorem 2.5. {x} is semi-closed set, since X is semi-T_1

Now x ϵ {x}⊆A.ThereforeA ϵ $\delta P_S O(X)$ by Proposition 3.2. Hence δPO(X) ⊆ $\delta P_S O(X)$.

But $\delta P_S O(X) \subseteq \delta PO(X)$ in general, by Proposition 3.2. Therefore, $\delta P_S O(X)$ = δPO(X).

Proposition 3.18. Every regular open set is δP_S-open set (ie)$RO(X) \subseteq \delta P_S O(X)$.

Proof: Let A be regular open by aTheorem 3.2[11],A is semiclosed and preopen.

Now A is preopen⇒A is δ-preopen [By Remark 2.21]

A is semiclosed ⇒ for each x ϵ A there exists the semi-closed set A itself, such that $x \in A \subseteq A$

Hence A is in $\delta P_S O(X)$.

$$\therefore RO(X) \subseteq \delta P_S O(X)$$

The converse of Proposition 3.18 is not true in $\delta P_S O(X)$.

Example 3.19. Let X = {a,b,c}, τ = {X, φ, {a}}. Then {b} ϵ $\delta P_S O(X)$ but {b}□RO(X)

Proposition 3.20. Every P_S-open set is a δP_S-open set.

Proof: Let A be a P_S-open set, By Lemma-2.20, A is preopen. ⟶ (1)

Moreover since A is P_S-open, we get, for each x ϵ A there exists a semi-closed set F such that

$x \in F \subseteq A$ ⟶ (2)

From (1) & (2) we get, A is δP_S-open.

Proposition 3.21. Each clopen set is δP_S-open.

Proof: The proof follows from Lemma 2.22 and Proposition 3.18.

Proposition 3.22: A δ-open set is a δP_S-open set.

Proof: From Theorem 2.8[4], A δ-open set is P_S-open set.

Now by above property, a P_S-open set is a δP_S-open set. Now by Proposition 3.20, a P_S-open set is δP_S-open set.

Hence a δ-open set is δP_S-open set.

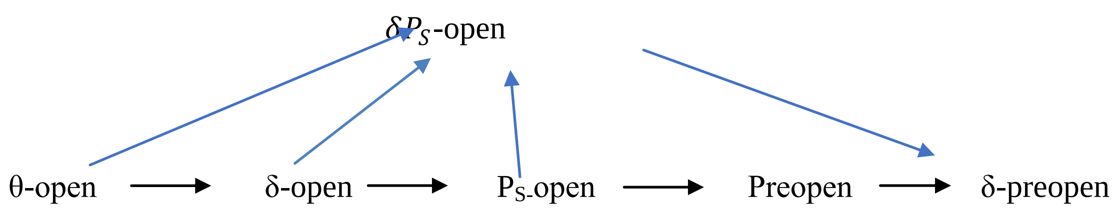
Proposition 3.23. Each θ-open set is δP_S-open set.

Proof: Every θ-open is regular open from Remark 2.21. Every regular open is δP_S-open by Proposition 3.18. Hence every θ-open set is δP_S-open set.

Note 3.24. α-open need not be δP_S -open set.

Consider example 3.4, {a} is α-open but not δP_S-open set.

Remark3.25. From all the above Propositions we have the following figure:



Remark 3.26. δP_S-open sets are independent with open sets.

Example 3.27. Consider X={a,b,c} with the topology$\tau = \{X, \phi, \{a\}\}$. Then {a} is open but not δP_S-open and {c} is δP_S-open but not open.

Lemma 3.28. In a hyperconnected space,

a) δO(X) = {X, φ}

b) δPO(X) = $\mathcal{P}(X)$

Proof: (a) Let (X,τ) be hyperconnected. Then for G ϵ τ, ClG = X ⟶ (1)

IfA≠ φ and A ϵ δO(X), for all x ϵ A, there exists an open set G such that x ϵ G ⊆ Int Cl G = Int X = X⊆A A =X. ∴ $\delta O(X) = \{X, \phi\}$

(b) For any subset A, A⊆X = Int X = Int (δCl A) [∵ $\delta C(A) = \{X, \varphi\}$]

A is δ-preopen. δPO(X) = $\mathcal{P}(X)$, the power set of X

Proposition 3.29. In a hyperconnected space,$SC(X) \subseteq \delta P_S O(X)$

Proof: LetX be hyperconnected. Then by lemma 3.28(ii) any subset is δ-preopen.

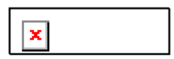
Let A ϵ SC(X). Now A is δ-preopen and semi-closed. HenceA ϵ $\delta P_S O(X)$.

Theorem 3.30. If$\delta P_S O(X) = \{X, \phi\}$ then (X, τ) is hyperconnected.

Proof: Suppose that$\delta P_S O(X) = \{X, \phi\}$. Since$RO(X) \subseteq \delta P_S O(X)$by Proposition 3.18, then $RO(X) = \{X, \phi\}$. By Lemma 2.10, we have (X, τ) is a hyperconnected space.

The converse is not true in general it can be seen from the following example.
Example 3.31. Let X ={a,b,c}, τ={X,φ,{a}}. Then (X, τ) is hyperconnected, since Cl{a}=X leading to RO(X,τ)={X,φ}. But$\delta P_S O(X) = \{X,\phi,\{b\},\{c\},\{b,c\}\} \neq \{X,\phi\}$.

Remark 3.32. It is to be noted that in the case of P_S-open sets, (X, τ) is hyperconnected if and only if $P_S O(X) = \{X,\phi\}$ which is not true in the case of δP_S-open sets.

Theorem 3.33. In a locally indiscrete space,$\delta P_S O(X) = \tau$.
Proof: Let (X, τ) be a locally indiscrete space. Let$U \subseteq X$, such that U ϵ τ. By definition 2.11 of locally indiscrete space every open set in X is closed, then IntClU=U which implies that U ϵ RO(X).By Proposition 3.18, U ϵ δP_SO(X). Thus τ ⊆ $\delta P_S O(X)$
Conversely, take V ϵ δP_SO(X). Then V is δ-preopen and for each x ϵ V and there exists a closed set F such that$x \in F \subseteq V$. By lemma 2.11, F is open making V open. Thus$\delta P_S O(X) \subseteq \tau$.
Proposition 3.34. In a locally indiscrete space, $SC(X) \subseteq \delta P_S O(X)$.
Proof: The proof is similar to that of Proposition 3.29.
Corollary 3.35. In a locally indiscrete space, a singleton {x} is semi-closed if and only if $\{x\} \subseteq \delta P_S O(X)$.
Proof: If {x] is semi-closed then by Proposition 3.34, {x} ϵ$\delta P_S O(X)$
Conversely, if {x} ϵ$\delta P_S O(X)$, there exists a semi-closed set F such that  which implies {x}=F. Hence {x} is semi-closed. Thus {x} ϵ SC(X).
Proposition 3.36. If A ϵ βO(X)∩P_SO(X), then A∈ $P_S O(X)$
Proof. LetA ϵ βO(X)∩P_SO(X). Then δ-preopen sets are the same as preopen sets from Theorem 2.17(b). Then δP_S-open sets are identical with P_S-open sets. Hence A ϵ δP_SO(X)
Proposition 3.37. If a topological space (X, τ) is s-regular, then τ⊆ δP_SO(X).
Proof: Let A⊆X and A ϵ τ.
If A = φ, thenA∈ $\delta P_S O(X)$
If$A \neq \varphi$,since X is s-regular, then by definition 2.3 for each x ϵ A, there exists U ϵ SO(X) such that x ϵ U⊆sCl U⊆A. Thus, we have x ϵ sCl U⊆A. Since A ϵ τ, we get A∈PO(X) which implies A ϵ δPO(X).
Moreover for all x ∈ A there exist a s-closed set sclU such that $x \in scl\ U \subseteq A$.
Hence A ϵ δP_SO(X). Thus τ⊆ δP_SO(X)
Proposition 3.38. For any topological space (X,τ), we have:

a) If τ (resp., δPO(X)) is indiscrete, then δP_SO(X) is also indiscrete.
b) IfδP_SO(X) is discrete, then δPO(X) is discrete.

Proof:case-(i) Let τ be indiscrete then τ = {X,φ}, since δO(X)⊑τ ,we get$\delta O(X) = \{X,\phi\}$ and $SO(X) = \{X,\phi\}$.Hence if A is δ-preopen and for each xϵA there is no semiclosed set except X containing x and contained in A. Hence$\delta P_S O(X) = \{X,\phi\} \Rightarrow \delta P_S O(X)$ is indiscrete.

case (ii) Even if $\delta PO(X)$ is indiscrete, then the result follows as in case – i, since $\delta P_S O(X) \subseteq \delta PO(X)$

(b)Follows from the fact that $\delta P_S O(X) \subseteq \delta PO(X)$

Note 3.39. In the case of P_S-open sets, τ is discrete if and only if $P_S O(X)$ is discrete. But in the case of δP_S-open sets, this property fails since τ and $\delta P_S O(X)$ are independent.

Proposition 3.40. Let (X, τ) be a space and $x \in X$. Then

a) If $\{x\} \in \delta P_S O(X)$, then $\{x\} \in SC(X)$.

b) $\{x\} \in \delta P_S O(X)$ if and only if $\{x\} \in RO(X)$.

Proof.(a) Let $\{x\} \in \delta P_S O(X)$. Then there exists a sclosed set F such that $x \in F \subseteq \{x\}, \Rightarrow \{x\}$ is semi-closed. Then $\{x\} \in SC(X)$.

(b) Let $\{x\} \in \delta P_S O(X) \Rightarrow \{x\}$ is semi-closed by(i).(i.e.,) $IntCl\{x\} \subseteq \{x\} \Rightarrow \{x\}$ is clopen $\Rightarrow \{x\} \in RO(X)$.

Converse part is proved in Theorem 3.18.

Corollary 3.41. For any subset $A \subseteq X$. The following conditions are equivalent:

a) A is clopen.
b) A is δ-open and δ-closed.
c) A is P_S-open and δ-closed.
d) A is α-open and δ-closed.
e) A is δP_S-open and δ-closed.
f) A is δ-preopen and δ-closed

Proof. $(a) \Rightarrow (b)$ By Lemma 2.7[5]

$(b) \Rightarrow (c)$ By Lemma 2.25

$(c) \Rightarrow (d)$ From (c) we get A is Ps-open and δ-closed. Now byCorollary 2.16[4]

A is α-open

$\therefore (c) \Rightarrow (d)$

$(d) \Rightarrow (e)$ From (d) A is α-open ⇒ A is δ-preopen ⟶ (1)

A is δ-closed ⇒ A is semi-closed. ⟶ (2)

(1) &(2) ⇒ A is δPs-open.

$\therefore (d) \Rightarrow (e)$

$(e) \Rightarrow (f)$ A is δPs-open ⇒δ-preopen by definition 3.1

$\therefore (e) \Rightarrow (f)$

$(f) \Rightarrow (a)$ Now A is δ-preopen $\Rightarrow A \subseteq int(\delta cl A)$

But A is δ-closed also $\therefore A \subseteq int\ A \Rightarrow A$ is open and

Moreover A is δ-closed⇒ A is closed

Hence A is clopen.

$$\therefore (f) \Rightarrow (a)$$

Corollary 3.42. For asemi-regular space, the following conditions are equivalent:

a) A is regular open.
b) A is P_S-open and semiclosed.
c) A is open and semiclosed.
d) A is α-open and semiclosed.
e) A is δP_S-open and semiclosed.
f) A is δ-preopen and semiclosed.

Proof. In a semi-regular space, δ-closed sets coincide with closed sets. So δPO(X)=PO(X) and$\delta P_S O(X) = P_S O(X)$.
Hence the result follows from Corollary 2.17[4].

Corollary3.43. For any topological space, the following statements are equivalent:

a) A is regular open.
b) A is δ-open and δ-semiregular.
c) A is δ-open and δ-semi-θ-closed.
d) A is δ-open and δ-semiclosed.
e) A is α-open and δ-semiclosed
f) A is δP_S-open and δ-semiclosed.
g) A is δ-preopen and δ-semiclosed.
h) A is α-preopen and e*closed.

Proof. Theproof follow from Theorem 7.15[29] anf from Corollary3.42

Proposition 3.44: For any space(X, τ), $\delta P_S O(X, \tau) = P_S O(X, \tau_s)$

Proof: By Lemma 2.24, $A \in \delta PO(X, \tau) \Leftrightarrow A \in PO(X, \tau_s)$

Proposition 3.45. For any topological space,if A ϵ δPO(X) and either A ϵ ηO(X)∪ SθO(X), then A ϵ $\delta P_S O(X)$.

Proof: Let A ϵ ηO(X) and A ϵ δPO(X). If $A = \phi$, then A ϵ $\delta P_S O(X)$. If$A \neq \phi$,Since A ϵ ηO(X), then A =∪ F_α, where$F_\alpha \in \delta C(X)$, for each α. Since$\delta C(X) \subseteq SC(X)$, then $F_\alpha \in SC(X)$, for each α. Since A ϵ δPO(X). Then by Proposition 3.2, $A \in \delta P_S O(X)$. Suppose that A ϵ SθO(X) and A ϵδPO(X). If X = φ, then A ϵ $\delta P_S O(X)$. IfA≠ φ, Since A ϵ SθO(X), then for each x ϵ A, there exists U ϵ SO(X) such that$x \in U \subseteq sClU \subseteq A$implies that $x \in sClU \subseteq A$and A ϵ δPO(X). Therefore, by Definition 3.1, A ϵ $\delta P_S O(X)$.

Corollary 3.46. For any subset A of a space X. If A∈ θSO(X) ∩δPO(X), then A∈$\delta P_S O(X)$.

Proof. Follows from the Proposition 3.45, and the fact that θSO(X) ⊂ SθO(X) or θSO(X) ⊂ ηO(X) [10].

Proposition 3.47. Let (X,τ) be any extremally disconnected space. If A ϵ θSO(X), then A ϵ $\delta P_S O(X)$.

Proof. Let A ϵ θSO(X). If A = φ, then A ϵ $\delta P_S O(X)$. If A ≠ φ. Since X is extremally disconnected, then by Theorem 2.8, θSO(X) = δO(X). Hence A ϵ δO(X). But δO(X)⊆$\delta P_S O(X)$ by Proposition 3.22. Therefore, A ϵ $\delta P_S O(X)$.

4.Subspace Properties in δP_S– Open Sets in Topological Spaces

Proposition4.1. Let (Y, τ_Y) be a subspace of a space (X, τ). If A ϵ $\delta P_S O(X, \tau)$ and $A \subseteq Y \ni Y \epsilon \delta PO(X)$, then A ϵ $\delta P_S O(Y, \tau_Y)$.

Proof. Let A ϵ $\delta P_S O(X, \tau)$, then A ϵ δPO(X, τ) and for each x ϵ A, there exists F ϵ SC(X, τ) such that x ϵ F ⊆A. Since A ϵ δPO(X,τ) and A ⊆ Y∋ $Y\epsilon\delta PO(X)$

Then by Lemma 2.27, A ϵ δPO(Y, τ_Y). Since F ϵ SC(X,τ) and F ⊆ Y. Then by Theorem-2.13(b), F ϵ SC(Y, τ_Y). Hence A ϵ $\delta P_S O(Y,\tau_Y)$.

Proposition4.2. Let (Y,τ_Y) be a subspace of a space (X,τ). If A ϵ $\delta P_S O(Y, \tau_Y)$ and Y ϵ RO(X, τ), then A ϵ $\delta P_S O(X, \tau)$.

Proof. Let A ϵ $\delta P_S O(Y,\tau_Y)$, then A ϵ δPO(Y,τ_Y) and for each x ϵ A, there exists F ϵ SC(Y,τ_Y) such that x ϵ F ⊆ A. Since Y ϵ RO(X,τ), then Y ϵ δPO(X,τ) and since A ϵ δPO(Y,τ_Y), then by Theorem2.18, A ϵ δPO(X,τ).

Again since Yϵ RO(X,τ), then Y ϵ SC(X,τ) and since F ϵ SC(Y, τ_Y), by Theorem 2.13(c), F ϵ SC(X,τ). Hence, A ϵ $\delta P_S O(X,\tau)$.

Corollary4.3. Let Y be a regular open subspace of a space X and let A be a subset of Y. Then A ϵ $\delta P_S O(Y)$ if and only if A ϵ $\delta P_S O(X)$.

Proof. Follows directly from Proposition4.1 and Proposition4.2.

Proposition4.4. Let A and B be any subsets of a space X. If A ϵ $\delta P_S O(X)$ and B ϵ RO(X), then A∩B ϵ $\delta P_S O(X)$.

Proof: Let A ϵ $\delta P_S O(X)$ and then A ϵ δPO(X) and A = ∪Fα where F_α ϵ SC(X) for each α,by Proposition 3.2, Then A ∩ B = (∪Fα) ∩ B = ∪ (Fα ∩ B). Since B ϵ RO(X), B is δ-open and since $A \in \delta PO(X)$by Theorem 2.14, A ∩ B ϵ δPO(X)⇒ $A \cap B \in \delta PO(B)$.Again since B ϵ RO(X), Bis open and Hence by Lemma 2.15 Fα ∩ B ϵ SC(B) for each α. ThusA ∩ B ϵ $\delta P_S O(B)$.

Proposition4.5. Let A and B be any subsets of a space X. If A ϵ $\delta P_S O(X)$ and B ϵ RSO(X), then A ∩ B ϵ $\delta P_S O(B)$.

Proof. Let A ϵ $\delta P_S O(X)$, then A ϵ δPO(X) and A = ∪Fα where Fα ϵ SC(X) for each α by Proposition 3.2. Then A ∩ B = (∪Fα) ∩ B = ∪ (Fα ∩ B). Since B ϵ RSO(X), then B ϵ δSO(X) and by Lemma 2.28, A ∩ B ϵ δPO(B).Again since B ϵ RSO(X), then B ϵ SC(X) and hence Fα ∩ B ϵ SC(X) for each α. Since Fα ∩ B⊆B and Fα ∩ B ϵ SC(X) for each α. Then by Theorem 2.13 (b), Fα ∩ B ϵ SC(B). Therefore, by Proposition 3.2, A ∩ B ϵ $\delta P_S O(B)$.

Proposition4.6. If A ϵ $\delta P_S O(X)$ and B is an δ-open subspace of a space X, then A ∩ B ϵ $\delta P_S O(B)$.

Proof. Let A ϵ $\delta P_S O(X)$, then A ϵ δPO(X) and A = ∪Fα where Fα ϵ SC(X) for each α by Proposition 3.2. Then A ∩ B = ∪Fα ∩ B = ∪(Fα ∩ B). Since B is an δ-open subspace of X, and by lemma 2.13, A ∩ B ϵ δPO(X).Again since B is δ-open then B is open. Then by Lemma 2.15, Fα ∩ B ϵ SC(B) for each α. Then by Proposition 3.2, A ∩ B ϵ $\delta P_S O(B)$.

Corollary4.7. If either B ϵ RSO(X) or B is an δ-open subspace of a space X and A ϵ $\delta P_S O(X)$, then A ∩ B ϵ $\delta P_S O(B)$.

Proof. Follows directly from Proposition4.5 and Proposition4.6.

Note 4.8. In the case of P_S-open sets, the above Proposition is true when B is open in X. But with δP_S-open sets, it must be modified to that B is a δ-open set.

REFERENCES

1. Abd EI-Monsef M. E., EI-Deeb S. N. and Mahmood R.A., β-open sets and β-continuous mappings, Bull. Fac. Sci. Assuit. Univ., 12(1)(1983), 1-18.
2. Abdulla A. S., On some applications of special subsets in topology, Ph.D. Thesis, Tanta Univ., 1986.
3. Ahmed N. K., On Some Types of Separation Axioms, M.Sc., Thesis, College of Science, Salahaddin Univ., 1990.
4. Alias B. Khalaf, and Baravan A. Asaad, P_S-open sets and P_S-Continuity in Topological spaces, J.Duhok Univ., 12(2)(2009), 183-192.
5. S.S.Benchalli et al., Contra δgb-Continuous Functions in Topological Spaces, European Journal of Pure And Applied Mathematics Vol. 10, No. 2, 2017, 312-322.
6. Caldas M., Fukutake T., Jafari S. and Noiri T., Some Applications of δ-preopen Sets in Topological Spaces, Bulletin of the Institute of Mathematics Academia Siniga 33(3) (2005), 261-276.
7. Di Maio G. And Noiri T., Di Maio G. and Noiri T., On s-closed spaces, Indian J. Pure Appl. Math., 18 (3) (1987), 226-233.
8. Dlaska K. and Ganster M., S-sets and co-S-closed topologies, Indian J. Pure Appl. Math., 23 (10) (1992), 731-73.
9. Dontchev J., Survey on preopen sets, The Proceedings of the Yatsushiro Topological Conference,(1998), 1-18.

10. Dontchev J., Popvassilev S. and Stavrova D., On the η-expansion topology for the co-semi-generalization and mildly Hausdorff spaces, Acta Math. Hungar., 80(1-2) (1998), 9-19.
11. Dontchev J. and Noiri T., Contra-semicontinuous functions, Math. Pannonica, 10 (1999), 159-168.
12. E. Ekici, One∗ -open sets and (D, S) ∗ -sets, Mathematica Moravica, 13 (2009), 29-36
13. El-Deeb S. N. Hasanein I. A, Mashhour A.S. and Noiri T., On P-regular spaces, Bull. Math. Soc. Sci. Math. R.S. Roum, 27(4)(1983), 311-315.
14. Guo T. Y., A characterization of extremally disconnected spaces, J. Central China Normal Univ. Natur. Sci., 21(2) (1981),169-170.
15. Jankovic D. S., A note on mappings of externally disconnected spaces, Acta math. Hungar., 46(1-2)(1985),83-92.
16. Joseph J. E. andKwack M. H., On S-closed spaces, Proc. Amer. Math. Soc., 80(2) (1980),341-348.
17. Levine N., Semi-open sets and semi-continuity in topological spaces, Amer. Math. Monthly, 70(1) (1963), 36-41.
18. Maheshwari S. N. and Prasad R., Some new separation axioms, Ann. Soc. Sci. Bruxelles, Ser. I., 89(1975), 395-402.
19. Masshour A. S., EI-Monsef M. E. A.,and EI-Deeb S.N., On precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc., Egypt, 53(1982), 47-53.
20. Masshour A. S., Hasanein M. E. A.,and EI-Deeb S.N., α-continuous and α-open mappings, Acta Mathematica Hungarica, 41(1983), 213-218.
21. Navalalagi G. B., Pre-neighbourhoods,The Mathematics Education, 32(4)(1998), 201-206.
22. Njastad O, On some classes of nearly open sets, Pacific J. Math. 15 (1965), 961– 970
23. Park J.H., Lee B.Y. and Son M.J., On δ-semiopen sets in topological spaces, J. Indian Acad. Math., 19, No 1 (1997), 59-67.
24. Ramprasad Paul and Bhattacharyya P, On Pre-Urysohn Spaces, Bulletin of the Malaysian Mathematical Society, 22(1999) 23-24.
25. Raychaudhuri S. and Mukherjee M. N., On δ-almost continuity and δ-preopen sets, Bull. Inst. Math. Acad. Sinica, 21 (1993), 357-366.
26. Stone M. H., Applications of the theory of Boolean rings to topology, Trans. Amer. Math. Soc., 41(1937), 375-481.
27. Velicko N. V., H-closed Topological spaces, Amer. Math. Soc. Transl., 78(2)(1968),103-118.
28. Yunis R. H., Properties of θ-semi-open sets, Zanco J. Of Pure and Applied Sciences, 19(1)(2001), 116-122.

29. Zanyar A. Ameen, Baravan A. Asaad and Ramadhan A. Muhammed, On superclasses of δ-open sets in topological spaces, Int. J. of Applied Mathematics, 32(2), (2019) 259-277.